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REDUCIBILITY AND NONREDUCIBILITY BETWEEN ℓ^p EQUIVALENCE RELATIONS

RANDALL DOUGHERTY AND GREG HJORTH

Abstract. We show that, for $1 \le p < q < \infty$, the relation of ℓ^p-equivalence between infinite sequences of real numbers is Borel reducible to the relation of ℓ^q-equivalence (i.e., the Borel cardinality of the quotient $\mathbb{R}^{\mathbb{N}}/\ell^p$ is no larger than that of $\mathbb{R}^{\mathbb{N}}/\ell^q$), but not vice versa. The Borel reduction is constructed using variants of the triadic Koch snowflake curve; the nonreducibility in the other direction is proved by taking a putative Borel reduction, refining it to a reduction map that is not only continuous but 'modular,' and using this nicer map to derive a contradiction.

0. Introduction

We start by recalling some background material on Borel equivalence relations; for more information, see [1].

A pair (X, B) is said to be a *standard Borel space* if there is a Polish topology on X — or here we may even demand that the topology be compact and metrizable — such that B is the Borel σ-algebra generated by the open sets. A *Borel* map $\theta\colon (X, B) \to (Y, C)$ between standard Borel spaces is one which pulls back elements of C to members of B. We may then say that two standard Borel spaces are *Borel isomorphic* if there is a Borel bijection between them with Borel inverse, though in fact it can be proved that any Borel bijection necessarily has a Borel inverse.

The classification problem for standard Borel spaces is trivial. Every standard Borel space has one of $\{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$ as its cardinality. Two standard Borel spaces are Borel isomorphic if and only if they have the same cardinality.

Frequently, though, we are led to consider the quotients of standard Borel spaces by Borel equivalence relations, and this leads to a classification problem that is nontrivial and of very great generality.

Let X and Y — or, more exactly, (X, B) and (Y, C) — be standard Borel spaces, and let E and F be Borel equivalence relations on the respective spaces. It is natural to say that Y/F has *Borel cardinality as great as* X/E, and write $E \le_B F$, if there is a Borel map $\theta\colon X \to Y$ with

$$\forall x, z \in X\, (xEz \iff \theta(x)F\theta(z)).$$

In other words, $E \le_B F$ indicates the existence of a Borel $\theta\colon X \to Y$ which induces a Borel injection $\theta'\colon X/E \to Y/F$.

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Similarly, we say that X/E and Y/F have the *same Borel cardinality* if $E \leq_B F$ and $F \leq_B E$, and the Borel cardinality of X/E is *strictly less than* that of Y/F (denoted $E <_B F$) if $E \leq_B F$ holds but $F \leq_B E$ fails.

The most obvious of all equivalence relations is the identity equivalence relation, and for a space X we use $\Delta(X)$ for the relation of equality on X, so that $x\Delta(X)z$ if and only if $x, z \in X$ and $x = z$. It is natural to identify X with $X/\Delta(X)$, and so the classification of Borel equivalence relations encompasses the classification of standard Borel spaces.

An important class of equivalence relations consists of those arising as the orbit equivalence relation induced by the Borel action of a Polish group. In the case that the group G acts on the space X, let E_G be the induced equivalence relation, so that for $x, z \in X$ we have xE_Gz if and only if there is some $g \in G$ with $g \cdot x = z$. We will write X/G for the quotient X/E_G, the class of all orbits.

Here the wolves come to us in the guise of sheep, and even apparently innocuous equivalence relations may lead to surprising increases in Borel cardinality, quickly drawing us away from that of X. As mentioned in [3], for the action of translation of $\mathbb{Q}$ on $\mathbb{R}$ we have

$$\mathbb{R}/\Delta(\mathbb{R}) <_B \mathbb{R}/\mathbb{Q};$$

in other words, there is Borel $\theta\colon \mathbb{R} \to \mathbb{R}$ such that

$$\forall x, y \in \mathbb{R}\ (x = z \iff \theta(x) - \theta(z) \in \mathbb{Q})$$

but there is *no* converse $\rho\colon \mathbb{R} \to \mathbb{R}$ so that

$$\forall x, y \in \mathbb{R}\ (x - z \in \mathbb{Q} \iff \rho(z) = \rho(x)).$$

Similarly we may consider the actions of c_0 and ℓ^p ($p \in [1, \infty)$) on the sequence space $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{N} = \{0, 1, 2, \dots\}$). As remarked in [4],

$$\mathbb{R}/\mathbb{Q} <_B \mathbb{R}^{\mathbb{N}}/c_0 \qquad \text{and} \qquad \mathbb{R}/\mathbb{Q} <_B \mathbb{R}^{\mathbb{N}}/\ell^p,$$

while $\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell^p$ have incomparable Borel cardinalities.

In this paper we complete the picture by showing that, for $1 \leq p < q < \infty$,

$$\mathbb{R}^{\mathbb{N}}/\ell^p <_B \mathbb{R}^{\mathbb{N}}/\ell^q.$$

Thus, the quotients $\mathbb{R}^{\mathbb{N}}/\ell^p$ give a natural example of a long chain of distinct Borel cardinalities.

1. Reduction

First, some matters of notation. For a finite or infinite sequence s, we use $s(n)$ to indicate the value of s on its nth coordinate. We write st to denote concatenation of s and t, and similarly for concatenation of infinitely many finite sequences to form an infinite sequence. Let $\operatorname{len}(s)$ be the length of the sequence s. If $0 \leq a \leq b \leq \operatorname{len}(s)$, then $s \upharpoonright [a, b)$ denotes the subsequence $\langle s(a), s(a+1), \dots, s(b-1)\rangle$ of s. Similarly, if s is an infinite sequence, then $s \upharpoonright [a, \infty)$ is the tail of s starting at a: $\langle s(a), s(a+1), \dots\rangle$.

If s is a sequence of real numbers, let $\|s\|_p$ denote its ℓ^p-norm $\left(\sum_i s(i)^p\right)^{1/p}$. Note that, if s is obtained by concatenating sequences s_j, then $\|s\|_p^p = \sum_j \|s_j\|_p^p$.

In this section, we show the following:

Theorem 1.1. *For $1 \leq p < q < \infty$, there is a Borel reduction of $\mathbb{R}^{\mathbb{N}}/\ell^p$ to $\mathbb{R}^{\mathbb{N}}/\ell^q$.*

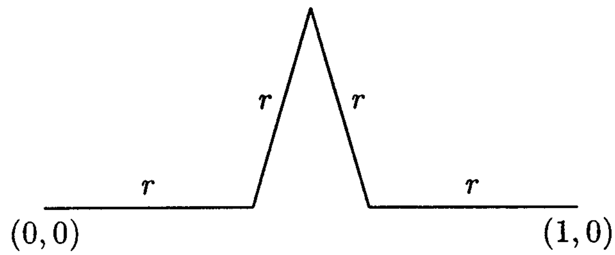

FIGURE 1. The generating polygon for the generalized Koch curve with parameter r.

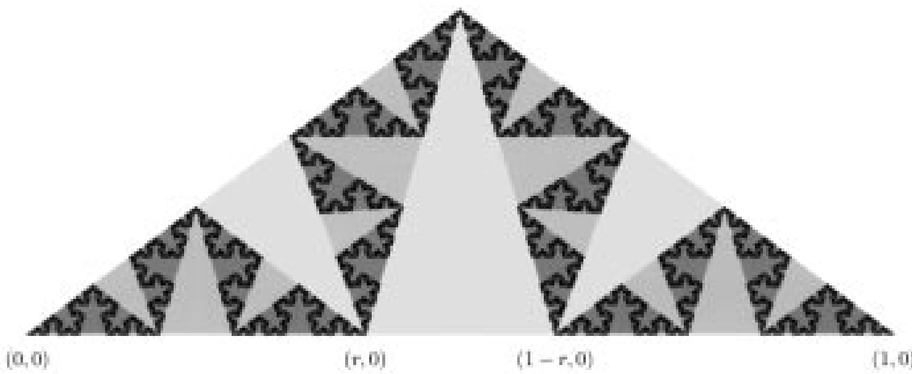

FIGURE 2. The generalized Koch snowflake curve K_r.

It will suffice to prove this theorem for the case where p and q are close to each other, specifically $1 \leq p < q < 2p$, because then we can use transitivity of $\leq_B$ (i.e., composition of reduction maps) to get the result for $1 \leq p < q < \infty$.

We will construct the required reduction by using a slightly generalized version of the Koch snowflake curve, described in any number of books on fractals (an appropriate primary reference is [6]). Fix a number r with $1/4 < r < 1/2$; the classic Koch curve uses $r = 1/3$. Start with a line segment of length 1, and then replace it with four segments of length r as shown in Figure 1.

Next, replace each new segment with a scaled-down copy of the generating polygon (matching up the endpoints). Repeat this infinitely many times; the resulting sequence of polygons will converge to a limit curve as shown in Figure 2. One can naturally parametrize this curve as a continuous map $K_r \colon [0,1] \to \mathbb{R}^2$ — the vertices of the generating polygon are $K_r(t)$ for $t = 0,\ 1/4,\ 1/2,\ 3/4,\ 1$, the vertices of the polygon after one scaling are $K_r(t)$ for t a multiple of $1/16$, and so on.

The following fact is well known, but we are not sure of the best reference for it, so we sketch the argument here. (A much more abstract and general version can be found in [2].)

Proposition 1.2 (folklore). *Suppose $1/4 < r < 1/2$, and let $K_r \colon [0,1] \to \mathbb{R}^2$ be the generalized Koch curve defined above. Let $\rho = -\log_4 r$. Then there are positive numbers m and M depending only on r such that, for all $x, y \in [0,1]$,*

$$m|x-y|^\rho \leq \|K_r(x) - K_r(y)\|_2 \leq M|x-y|^\rho.$$

Proof. This comes as a by-product of the usual proof that the Koch curve is a simple arc (given in [8, §13.7], for instance). The generating polygon of the curve is included in the closed triangular region shown in Figure 2. When this polygon is scaled down for the first time, one gets four scaled-down triangles as well; these four smaller triangles are included in the larger triangle, and they do not overlap except for one shared vertex for each consecutive pair in the list of four. After n scalings, one has a chain of 4^n triangles, overlapping only at shared vertices between consecutive triangles in the chain, which includes the approximating curve for that stage and all following stages, and hence includes the limit curve. In fact, each part $K_r \restriction [(k-1)/4^n, k/4^n]$ is included within the kth triangle in the chain. Note that the base of each such triangle has length r^n.

Given distinct x, y in $[0,1]$, there is a least number n such that x and y lie in disjoint (i.e., nonadjacent) intervals $[(k-1)/4^n, k/4^n]$ and $[(i-1)/4^n, i/4^n]$. Clearly $n \geq 1$. We have $|x-y| \geq 1/4^n$; on the other hand, since x and y lie in the same interval or adjacent intervals of length $1/4^{n-1}$, we have $|x-y| \leq 2/4^{n-1}$.

Now, the points $K_r(x)$ and $K_r(y)$ lie in the same triangle or adjacent triangles in the chain at stage $n-1$. The base of each such triangle has length r^{n-1}, and no two points in the triangle are farther away from each other than the two base vertices (because the base angles are less than $\pi/4$). Therefore, we must have $\|K_r(x) - K_r(y)\|_2 \leq 2r^{n-1}$. On the other hand, $K_r(x)$ and $K_r(y)$ lie in triangles in the chain at stage n which are not adjacent but are close to each other in the chain (there are at most six other triangles between them in the chain). Since there are only two different angles at which adjacent triangles in the chain can meet, there are only finitely many possible relative positions for two triangles in the chain which are not adjacent but are within six triangles of each other. (The possible relative configurations are scaled by r^n but are otherwise independent of n.) Therefore, we can find a number $c > 0$ depending on r but not on n such that two such triangles have to lie at a distance at least cr^n from each other. This gives $\|K_r(x) - K_r(y)\|_2 \geq cr^n$.

Using

$$1/4^n \leq |x-y| \leq 2/4^{n-1} = 8/4^n,$$

we get

$$(1/4)^{\rho n} \leq |x-y|^\rho \leq 8^\rho (1/4)^{\rho n}.$$

But $(1/4)^{\rho n}$ is just r^n. Therefore, we have

$$m|x-y|^\rho \leq \|K_r(x) - K_r(y)\|_2 \leq M|x-y|^\rho,$$

where $m = c/8^\rho$ and $M = 2/r$. □

(Note: The similarity dimension of K_r is $1/\rho$.)

We now extend K_r to a continuous function $\bar{K}_r$ from all of $\mathbb{R}$ to $\mathbb{R}^2$ by laying copies of K_r end-to-end: for any integer i and any $t \in [i, i+1]$, let $\bar{K}_r(t) = K_r(t-i) + (i, 0)$. Then $\bar{K}_r$ will have almost the same properties as K_r has. If x and y are in the same interval $[i, i+1]$, then the inequalities of Proposition 1.2 hold for $\|\bar{K}_r(x) - \bar{K}_r(y)\|_2$. If x and y are not in the same such interval but are in adjacent intervals, say

$$i-1 \leq x \leq i \leq y \leq i+1,$$

then the fact that the triangle in Figure 2 has base angles less than $\pi/4$ implies that

$$\begin{aligned}&\max(\|\bar{K}_r(x)-\bar{K}_r(i)\|_2, \|\bar{K}_r(i)-\bar{K}_r(y)\|_2)\\&\quad\le \|\bar{K}_r(x)-\bar{K}_r(y)\|_2 \le \|\bar{K}_r(x)-\bar{K}_r(i)\|_2+\|\bar{K}_r(i)-\bar{K}_r(y)\|_2;\end{aligned}$$

since

$$\max(|x-i|^\rho, |i-y|^\rho) \ge |(x-y)/2|^\rho = 2^{-\rho}|x-y|^\rho$$

and

$$|x-i|^\rho + |i-y|^\rho \le 2|x-y|^\rho,$$

we get

$$m'|x-y|^\rho \le \|\bar{K}_r(x)-\bar{K}_r(y)\|_2 \le M'|x-y|^\rho$$

with $m' = m/2^\rho$ and $M' = 2M$. Finally, if x and y are in non-adjacent intervals $[i, i+1]$, then $|x-y| \ge 1$ and $\|\bar{K}_r(x)-\bar{K}_r(y)\|_2 \ge 1$.

(Alternatively, one could define an extension of K_r to all of $\mathbb{R}$ by using self-similarities of ratio r^{-1} just as K_r was defined to satisfy self-similarities with ratio r. This would require more work in the definition, but it would give an extended curve satisfying the inequalities of Proposition 1.2 without change for all $x, y \in \mathbb{R}$.)

Proof of Theorem 1.1. As noted earlier, we may assume $1 \le p < q < 2p$. Let $\rho = p/q$ and $r = 4^{-\rho}$; then we have $\rho = -\log_4 r$, $1/2 < \rho < 1$, and $1/4 < r < 1/2$. Define the mapping $\theta\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ as follows: given $x = \langle x(0), x(1), \ldots\rangle$, obtain $\theta(x)$ by concatenating $\bar{K}_r(x(j))$ for $j = 0, 1, \ldots$ (so each coordinate of x yields two coordinates of $\theta(x)$). Clearly θ is continuous; we will show that, for all $x, y \in \mathbb{R}^\mathbb{N}$, $x - y \in \ell^p$ if and only if $\theta(x)-\theta(y) \in \ell^q$.

Standard computations show that there are positive constants c and C such that, for any $w \in \mathbb{R}^2$,

$$c\|w\|_2 \le \|w\|_q \le C\|w\|_2;$$

in fact,

$$2^{-1/2}\|w\|_2 \le \|w\|_\infty \le \|w\|_q \le \|w\|_1 \le 2\|w\|_2.$$

Therefore, we have

$$\begin{aligned}\theta(x)-\theta(y)\in\ell^q &\iff \|\theta(x)-\theta(y)\|_q^q < \infty\\ &\iff \sum_j \|\bar{K}_r(x(j))-\bar{K}_r(y(j))\|_q^q < \infty\\ &\iff \sum_j \|\bar{K}_r(x(j))-\bar{K}_r(y(j))\|_2^q < \infty.\end{aligned}$$

Consider two cases. First, suppose that there are infinitely many j such that $x(j)$ and $y(j)$ are in non-adjacent intervals of the form $[i, i+1]$, $i \in \mathbb{Z}$. For such j, we have $|x(j)-y(j)| \ge 1$ and $\|\bar{K}_r(x(j))-\bar{K}_r(y(j))\|_2 \ge 1$; hence, $x - y \notin \ell^p$ and $\theta(x)-\theta(y) \notin \ell^q$.

Now suppose the contrary; that is, there is a natural number N such that, for all $j > N$, x and y are in the same interval $[i, i+1]$ or in adjacent such intervals. Hence, for $j > N$, we have

$$m'|x(j)-y(j)|^\rho \le \|\bar{K}_r(x(j))-\bar{K}_r(y(j))\|_2 \le M'|x(j)-y(j)|^\rho;$$

therefore,

$$\begin{aligned}\theta(x) - \theta(y) \in \ell^q &\iff \sum_{j>N} \|\bar{K}_r(x(j)) - \bar{K}_r(y(j))\|_2^q < \infty \\ &\iff \sum_{j>N} (|x(j) - y(j)|^\rho)^q < \infty \\ &\iff \sum_{j>N} |x(j) - y(j)|^p < \infty \\ &\iff x - y \in \ell^p,\end{aligned}$$

as desired. □

2. Non-reduction

We use the same notations concerning sequences and ℓ^p as in the preceding section.

Definition 2.1. For $p \in [1, \infty)$ and $\vec{\epsilon} = \langle \epsilon_i \rangle_{i \in \mathbb{N}} \in \ell^p$, let $\mathbb{Z}(\vec{\epsilon})$ be the set of all $x \in \mathbb{R}^{\mathbb{N}}$ such that $x(n)$ is an integer multiple of ϵ_n for all $n \in \mathbb{N}$, and let $\mathbb{Z}_p(\vec{\epsilon}) = \mathbb{Z}(\vec{\epsilon}) \cap \ell^p$.

Note that $\mathbb{Z}_p(\vec{\epsilon})$ is a closed subgroup of ℓ^p. We have $\mathbb{Z}(\vec{\epsilon})/\mathbb{Z}_p(\vec{\epsilon}) \leq_B \mathbb{R}^{\mathbb{N}}/\ell^p$, since the inclusion map witnesses the Borel reduction. Also, $\mathbb{Z}(\vec{\epsilon})$ and $\mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ are Polish spaces, and thus in particular satisfy the Baire category theorem. They are more discrete versions of the spaces $\mathbb{R}^{\mathbb{N}}$ and $[0,1]^{\mathbb{N}}$. We can take as basic open sets for $\mathbb{Z}(\vec{\epsilon})$ the sets

$$N_s = \{x \in \mathbb{Z}(\vec{\epsilon}) \colon x = sy \text{ for some } y\}$$

where s is a finite sequence such that $s(i)$ is an integer multiple of ϵ_i for each i; the same applies to $\mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$.

Theorem 2.2. *For $1 \leq p < q < \infty$, there is no Borel reduction of $\mathbb{R}^{\mathbb{N}}/\ell^q$ to $\mathbb{R}^{\mathbb{N}}/\ell^p$.*

Proof. Suppose there is such a Borel reduction. Let $\vec{\epsilon} = \langle \epsilon_i \rangle_{i \in \mathbb{N}}$ be defined by $\epsilon_i = 2^{-i}$; then we have $\vec{\epsilon} \in \ell^q$, and the sequence which is constantly 1 is in $\mathbb{Z}(\vec{\epsilon})$. We may restrict the given Borel reduction to obtain a Borel function $\theta \colon \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that $\hat{x} - x \in \ell^q$ if and only if $\theta(\hat{x}) - \theta(x) \in \ell^p$.

We now reorganize θ along the general lines of [7] to obtain a Borel map which is not only continuous but 'modular,' meaning that the sequences produced by the function consist of finite blocks, each of which depends on only a single coordinate of the argument to the function. The first step is to find a suitable subset of $\mathbb{Z}(\vec{\epsilon})$ on which θ is continuous and 'almost modular.'

Claim (i). For any $j, k \in \mathbb{N}$, there exist $l \in N$, a finite sequence s^* with

$$s^*(i) \in \{0, \epsilon_{k+i}, 2\epsilon_{k+i}, \ldots, 1\} \text{ for all } i < \operatorname{len}(s^*),$$

and a comeager set $D \subseteq \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ such that, for all $x, \hat{x} \in D$, if we have $x = rs^*y$ and $\hat{x} = \hat{r}s^*y$ for some $r, \hat{r} \in \mathbb{R}^k$ and $y \in \mathbb{R}^{\mathbb{N}}$, then

$$\|(\theta(x) - \theta(\hat{x})) \restriction [l, \infty)\|_p < 2^{-j}.$$

Proof. For each $l \in \mathbb{N}$, define the function $F_l \colon \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}} \to \mathbb{R}$ by

$$F_l(x) = \max_{z,\hat{z}} \|(\theta(z) - \theta(\hat{z})) \restriction [l,\infty)\|_p,$$

where z and $\hat{z}$ are elements of $\mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ such that $z(i) = \hat{z}(i) = x(i)$ for all $i \geq k$. There are only finitely many such pairs $z, \hat{z}$, and for each such pair we have $z - \hat{z} \in \ell^q$, so $\theta(z) - \theta(\hat{z}) \in \ell^p$, so

$$\lim_{l\to\infty} \|(\theta(z) - \theta(\hat{z})) \restriction [l,\infty)\|_p = 0;$$

hence, $F_l(x) < \infty$ for all l and $\lim_{l\to\infty} F_l(x) = 0$. Therefore, by the Baire Category Theorem, there exists an l such that $\{x \colon F_l(x) < 2^{-j}\}$ is not meager. This set has the property of Baire, so there is a nonempty open set O on which it is relatively comeager.

We may take O to be a basic open set N_t for some finite sequence t, and we may assume $\text{len}(t) \geq k$. Write t as r^*s^* where $\text{len}(r^*) = k$. But $F_l(x)$ does not depend on the first k coordinates of x, so $\{x \colon F_l(x) < 2^{-j}\}$ is also relatively comeager in N_{rs^*} for all other r of length k. Let D be a comeager set such that $F_l(x) < 2^{-j}$ whenever $x \in D \cap N_{rs^*}$ for any r of length k. Now the conclusion of the claim follows from the definition of F_l. □

Since any Borel function is continuous on a comeager set [5, (8.38)], we can fix a dense G_δ set $C \subseteq \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ on which θ is continuous.

Claim (ii). For any $j, k, l \in \mathbb{N}$, there exists a finite sequence s^{**} with

$$s^{**}(i) \in \{0, \epsilon_{k+i}, 2\epsilon_{k+i}, \ldots, 1\} \text{ for all } i < \text{len}(s^{**})$$

such that, for all $x, \hat{x} \in C$, if we have $x = rs^{**}y$ and $\hat{x} = rs^{**}\hat{y}$ for some $r \in \mathbb{R}^k$ and $y, \hat{y} \in \mathbb{R}^{\mathbb{N}}$, then

$$\|(\theta(x) - \theta(\hat{x})) \restriction [0,l)\|_p < 2^{-j}.$$

Furthermore, if G is a given dense open subset of $\mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$, then s^{**} can be chosen such that $N_{rs^{**}} \subseteq G$ for all $r \in \mathbb{R}^k$ such that $r(i) \in \{0, \epsilon_i, 2\epsilon_i, \ldots, 1\}$ for $i < k$.

Proof. There are only finitely many such r's; list them as $r_0, r_1, \ldots, r_{M-1}$. We will build s^{**} by successive extensions.

Let t_0 be the empty sequence. Now suppose that $m < M$ and we have a finite sequence t_m, with $t_m(i) \in \{0, \epsilon_{k+i}, 2\epsilon_{k+i}, \ldots, 1\}$ for all $i < \text{len}(t_m)$. The basic open set $N_{r_m t_m}$ must meet the comeager set C, so choose $w \in C \cap N_{r_m t_m}$. Since θ is continuous on C, we can find an even smaller basic open neighborhood O of w such that, for all $x, \hat{x} \in C \cap O$, $\|(\theta(x) - \theta(\hat{x})) \restriction [0,l)\|_p < 2^{-j}$. This O must be of the form $N_{r_m t'_m}$ for some extension t'_m of t_m. Now we can extend t'_m further to get t_{m+1} such that $N_{r_m t_{m+1}} \subseteq G$, since G is open dense.

Once all of the sequences t_m are constructed, the final sequence t_M will be the desired s^{**}. □

We now repeatedly apply Claims (i) and (ii) to define natural numbers $b_0 < b_1 < b_2 < \cdots$ and $l_0 < l_1 < l_2 < \cdots$, finite sequences $s_0, s_1, s_2, \ldots$, and dense open sets $D_i^j \subseteq \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ $(i, j \in N)$ as follows. Let $b_0 = l_0 = 0$. Suppose we have b_j, l_j, and $D_i^{j'}$ for $j' < j$. Apply Claim (i) for this j with $k = b_j + 1$ to get a natural number l_{j+1}, a finite sequence s_j^*, and a comeager set D^j satisfying the conclusion of that claim; we may assume that $l_{j+1} > l_j$ and $D^j \subseteq C$. Let $D^j_0 \supseteq D^j_1 \supseteq D^j_2 \supseteq \cdots$ be dense open subsets of $\mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ whose intersection is included in D^j. Now apply Claim (ii) with the current j, $k = b_j + 1 + \mathrm{len}(s^*_j)$, $l = l_{j+1}$, and $G = \bigcap_{j'=0}^{j} D^{j'}_j$, to get another finite sequence s^{**}_j. Let $s_j = s^*_j s^{**}_j$ and $b_{j+1} = b_j + \mathrm{len}(s_j) + 1$.

Let C' be the set of all $x \in \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ of the form

$$\langle a_0 \rangle s_0 \langle a_1 \rangle s_1 \langle a_2 \rangle s_2 \cdots$$

where $a_j \in \{0, \epsilon_{b_j}, 2\epsilon_{b_j}, \ldots, 1\}$. In other words, coordinates $b_0, b_1, \ldots$ of a member of C' are unrestricted, but the remaining coordinates are specified by the sequences s_j.

Since $s_j = s^*_j s^{**}_j$, any $x \in C'$ has the form $r s^*_j y$ where r has length $b_j + 1$, and also has the form $r s^{**}_j y$ where r has length $b_j + \mathrm{len}(s^*_j) + 1$. Therefore, Claim (ii) for s^{**}_j gives $x \in \bigcap_{j'=0}^{j} D^{j'}_j$. Hence, for any j, we have $x \in D^j_i$ for all $i \geq j$, so $x \in D^j$. It follows that $x \in C$ as well. Therefore, Claims (i) and (ii) imply that, for any $x, \hat{x} \in C'$:

(1) if $x(b_i) = \hat{x}(b_i)$ for all $i \geq j+1$, then $\|(\theta(x) - \theta(\hat{x})) \restriction [l_{j+1}, \infty)\|_p < 2^{-j}$;
(2) if $x(b_i) = \hat{x}(b_i)$ for all $i \leq j$, then $\|(\theta(x) - \theta(\hat{x})) \restriction [0, l_{j+1})\|_p < 2^{-j}$.

Next, we strip off the unnecessary coordinates. Let $\vec{\epsilon}' = \langle \epsilon'_i \rangle_{i \in \mathbb{N}}$ be defined by $\epsilon'_i = \epsilon_{b_i} = 2^{-b_i}$. Then we can define a map $g \colon \mathbb{Z}(\vec{\epsilon}') \cap [0,1]^{\mathbb{N}} \to \mathbb{Z}(\vec{\epsilon}) \cap [0,1]^{\mathbb{N}}$ by

$$g(x) = \langle x(0) \rangle s_0 \langle x(1) \rangle s_1 \langle x(2) \rangle s_2 \cdots .$$

Clearly $g(x) \in C'$. Since $g(x)$ and $g(\hat{x})$ differ only on the coordinates copied over from x and $\hat{x}$, we have $\|g(x) - g(\hat{x})\|_q = \|x - \hat{x}\|_q$, so in particular $g(x) - g(\hat{x}) \in \ell^q$ iff $x - \hat{x} \in \ell^q$.

We can now construct the 'modular' Borel reducing map. For any $x \in \mathbb{Z}(\vec{\epsilon}')$ and $j \in \mathbb{N}$, let $e_j(x) \in \mathbb{Z}(\vec{\epsilon}')$ be defined by $e_j(x)(j) = x(j)$ and $e_j(x)(i) = 0$ for all $i \neq j$. Now define $\theta' \colon \mathbb{Z}(\vec{\epsilon}') \cap [0,1]^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by: for all j and all m such that $l_j \leq m < l_{j+1}$, let $\theta'(x)(m) = \theta(g(e_j(x)))(m)$. Then we actually have

$$\theta'(x) = f_0(x(0)) f_1(x(1)) f_2(x(2)) \cdots ,$$

where $f_j \colon \{0, \epsilon'_j, 2\epsilon'_j, \ldots, 1\} \to \mathbb{R}^{l_{j+1} - l_j}$ is given by

$$f_j(a) = \theta(g(\langle 0, 0, \ldots, 0, a, 0, 0, \ldots \rangle)) \restriction [l_j, l_{j+1}),$$

with j 0's before the a. It follows that

$$\|\theta'(x) - \theta'(\hat{x})\|_p^p = \sum_{j \in \mathbb{N}} \|f_j(x(j)) - f_j(\hat{x}(j))\|_p^p$$

for all $x, \hat{x} \in \mathbb{Z}(\vec{\epsilon}') \cap [0,1]^{\mathbb{N}}$.

Claim (iii). For all $x, \hat{x} \in \mathbb{Z}(\vec{\epsilon}') \cap [0,1]^{\mathbb{N}}$,

$$x - \hat{x} \in \ell^q \iff \theta'(x) - \theta'(\hat{x}) \in \ell^p.$$

Proof. It will suffice to show that $\theta'(x) - \theta(g(x)) \in \ell^p$ for all $x \in \mathbb{Z}(\vec{\epsilon}') \cap [0,1]^{\mathbb{N}}$, since this implies that

$$\begin{aligned} \theta'(x) - \theta'(\hat{x}) \in \ell^p &\iff \theta(g(x)) - \theta(g(\hat{x})) \in \ell^p \\ &\iff g(x) - g(\hat{x}) \in \ell^q \iff x - \hat{x} \in \ell^q. \end{aligned}$$

Let $e'_j(x) \in \mathbb{Z}(\vec{\epsilon}')$ be defined by $e'_j(x)(i) = x(i)$ for $i \le j$ and $e'_j(x)(i) = 0$ for $i > j$. Then $g(x)$ and $g(e'_j(x))$ agree on all coordinates below b_{j+1}, so (2) gives

$$\|(\theta(g(x)) - \theta(g(e'_j(x)))) \restriction [0, l_{j+1})\|_p < 2^{-j},$$

while $g(e'_j(x))$ and $g(e_j(x))$ agree on all coordinates above b_{j-1}, so (1) for $j-1$ gives

$$\|(\theta(g(e'_j(x))) - \theta(g(e_j(x)))) \restriction [l_j, \infty)\|_p < 2 \cdot 2^{-j}.$$

(If $j = 0$, then $e'_j(x) = e_j(x)$, so the above formula still holds.) Now the triangle inequality gives

$$\begin{aligned}\|(\theta(g(x)) - \theta'(x)) \restriction [l_j, l_{j+1})\|_p &= \|(\theta(g(x)) - \theta(g(e_j(x)))) \restriction [l_j, l_{j+1})\|_p \\ &\le \|(\theta(g(x)) - \theta(g(e'_j(x)))) \restriction [l_j, l_{j+1})\|_p \\ &\quad + \|(\theta(g(e'_j(x))) - \theta(g(e_j(x)))) \restriction [l_j, l_{j+1})\|_p \\ &\le \|(\theta(g(x)) - \theta(g(e'_j(x)))) \restriction [0, l_{j+1})\|_p \\ &\quad + \|(\theta(g(e'_j(x))) - \theta(g(e_j(x)))) \restriction [l_j, \infty)\|_p \\ &\le 3 \cdot 2^{-j}.\end{aligned}$$

Therefore,

$$\|(\theta(g(x)) - \theta'(x))\|_p^p = \sum_j \|(\theta(g(x)) - \theta'(x)) \restriction [l_j, l_{j+1})\|_p^p \le \sum_j 3^p 2^{-jp} < \infty,$$

as desired. □

Finally, we proceed toward a contradiction.

Claim (iv). There exist positive numbers $c \in \mathbb{R}$ and $N \in \mathbb{N}$ such that, for all $j > N$, $\|f_j(1) - f_j(0)\|_p > c$.

Proof. If not, then we can find natural numbers $j_0 < j_1 < j_2 < \cdots$ such that $\|f_{j_m}(1) - f_{j_m}(0)\|_p \le 2^{-m}$. Let $\hat{x}$ be the sequence which is constantly 0, and let x be the sequence which is 1 at all coordinates j_m ($m \in \mathbb{N}$) and 0 at all other coordinates. Then $x - \hat{x} \notin \ell^q$ but

$$\begin{aligned}\|\theta'(x) - \theta'(\hat{x})\|_p^p &= \sum_j \|f_j(x(j)) - f_j(\hat{x}(j))\|_p^p \\ &= \sum_m \|f_{j_m}(1) - f_{j_m}(0)\|_p^p \le \sum_m 2^{-mp} < \infty,\end{aligned}$$

contradicting Claim (iii). □

Choose positive numbers $\delta_0, \delta_1, \delta_2, \ldots$ such that δ_j is a multiple of ϵ'_j, 1 is a multiple of δ_j (say $1 = k_j\delta_j$), $\sum_j \delta_j^p = \infty$, and $\sum_j \delta_j^q < \infty$. Here is one way to do this: For each j, let d_j be the largest integer less than or equal to $(1/p)\log_2(j+1)$; note that $0 \le d_j \le j \le b_j$. Now let $\delta_j = 2^{-d_j}$. Then we have

$$\delta_j^p \ge (j+1)^{-1} \qquad \text{and} \qquad \delta_j^q \le 2^q(j+1)^{-q/p},$$

so $\sum_j \delta_j^p$ diverges but $\sum_j \delta_j^q$ converges, since $q/p > 1$.

For all sufficiently large j, we have $\|f_j(1) - f_j(0)\|_p > c$ by Claim (iv); so, since $k_j\delta_j = 1$, the triangle inequality gives

$$\|f_j(\delta_j) - f_j(0)\|_p + \|f_j(2\delta_j) - f_j(\delta_j)\|_p + \cdots + \|f_j(k_j\delta_j) - f_j((k_j-1)\delta_j)\|_p > c.$$

Therefore, there must be a natural number $n_j < k_j$ such that

$$\|f_j((n_j+1)\delta_j) - f_j(n_j\delta_j)\|_p > c/k_j = c\delta_j.$$

For j below the bound N given by Claim (iv), let $n_j = 0$. Now define $x, \hat{x} \in \mathbb{Z}(\vec{\epsilon}') \cap [0,1]^{\mathbb{N}}$ by $x(j) = (n_j+1)\delta_j$ and $\hat{x}(j) = n_j\delta_j$. Then we have

$$\|x - \hat{x}\|_q^q = \sum_j \delta_j^q < \infty$$

but

$$\|\theta'(x) - \theta'(\hat{x})\|_p^p = \sum_j \|f_j(x(j)) - f_j(\hat{x}(j))\|_p^p \geq \sum_{j>N} c^p \delta_j^p = \infty.$$

This contradicts Claim (iii), so we are done. □

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References

1. H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Notes Series, 232, Cambridge University Press, Cambridge, 1996. MR **98d:**54068
2. K. J. Falconer and D. T. Marsh, *Classification of quasi-circles by Hausdorff dimension*, Nonlinearity **2** (1989), 489–493. MR **91a:**58108
3. L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), 903–928. MR **91h:**28023
4. G. Hjorth, *Actions by the classical Banach spaces*, preprint, UCLA, 1996.
5. A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR **96e:**03057
6. H. von Koch, *On a continuous curve without tangents constructible from elementary geometry*, Classics on fractals (G. Edgar, ed.), Addison-Wesley, Reading, Massachusetts, 1993, pp. 25–45.
7. A. Louveau and B. Velickovic, *A note on Borel equivalence relations*, Proc. Amer. Math. Soc. **120** (1994), 255–259. MR **94f:**54076
8. C. Tricot, *Curves and fractal dimension*, Springer-Verlag, New York, 1995. MR **95i:**28005

Department of Mathematics, Ohio State University, Columbus, Ohio 43210
E-mail address: `rld@math.ohio-state.edu`

Department of Mathematics, University of California, Los Angeles, California 90095-1555
E-mail address: `greg@math.ucla.edu`